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Abstract

In this paper we show that BIBD(v, b, r, k, λ) where $v = pq$ or $pq + 1$, when written in the notation of Bose's method of differences may often be used to find generalized Bhaskar Rao designs GBRD($p, b', r', k, \lambda; G$) where G is a group of order q and vice versa. This gives many new GBRDs including a GBRD($9, 5, 5; Z_5$) and a GBRD($13, 7, 7; Z_7$).

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Bose's Method of Differences Applied to Construct Bhaskar Rao Designs

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R.C.Bose Memorial Conference on Statistical design and Related Combinatorics

Abstract

In this paper we show that $BIBD(v, b, r, k, \lambda)$, where $v = pq$ or $pq + 1$, when written in the notation of Bose's method of differences may often be used to find generalized Bhaskar Rao designs $GBRD(p, b', r', k, \lambda; G)$ where G is a group of order q and vice versa.

This gives many new GBRDs including a $GBRD(9, 5, 5; Z_5)$ and a $GBRD(13, 7, 7; Z_7)$.

1 Introduction

Let $G = \{h_1 = e, h_2, \dots, h_g\}$ be a finite group of order g with identity e . Form the matrix W

$$W = h_1 A_1 + \dots + h_g A_g$$

where A_1, \dots, A_g are $v \times b$ $(0,1)$ -matrices such that the Hadamard product $A_k * A_j = 0$ for any $k \neq j$. Let

$$W^+ = (h_1^{-1} A_1 + \dots + h_g^{-1} A_g)^T$$

and

$$N = A_1 + A_2 + \dots + A_g.$$

Then we say W is a *generalized Bhaskar Rao design* over G denoted by $GBRD(v, b, r, k, \lambda; G)$, or in abbreviated form $GBRD(v, k, \lambda; G)$, if N satisfies

$$NN^T = (r - \lambda)I + \lambda J :$$

that is, N is the incidence matrix of the $BIBD(v, k, \lambda)$ and

$$WW^+ = reI + (\lambda/g)(h_1 + \dots + h_g)(J - I).$$

We say that the design W is based on the matrix N or that N is signed over the group G .

$GBRD$ have been studied by the author and others, for example see [7, 8, 9, 14, 16, 21, 22]. We now illustrate the construction of the $GBRD$ with some examples. For further examples the reader is referred to [20].

To check whether a set of blocks $(x_a, y_b, z_c) \bmod (v, g)$, each with k elements, are part of a *GBRD* or a *BIBD* we check all the differences: $(y - x)_{ba^{-1}}^i, (x - y)_{ab^{-1}}^i, (z - x)_{ca^{-1}}^i, (x - z)_{ac^{-1}}^i, (z - y)_{cb^{-1}}^i$ and $(y - z)_{bc^{-1}}^i$, where the subscripts such as ab^{-1} are from a group G of order $|G| = g$ and the subscripted elements such as $(x - y)^i$ come from a group V of order v .

If each non-zero-zero difference, that is differences other than 0_0 , including the differences $0_1, \dots, 0_{g-1}$ occurs the same number of times, λ , we will have a *BIBD*(v, k, λ). If each non-zero difference, that is not including the differences $0_0, 0_1, \dots, 0_{g-1}$, occurs the same number of times, λ , we will have a *GBRD*($v, k, \lambda; G$).

We also use a distinguished element ∞ which has the properties that

$$\infty - a_i = \infty_i^{-1}, \text{ and } a_i - \infty = -\infty_i,$$

for any $a_i \in V$.

For a Bhaskar Rao design ∞_j and $-\infty_j$ must occur λ times. For a *BIBD* it is merely necessary that the number of elements in the sets with ∞ is $k - 1$.

Example 1 Consider the initial blocks

$$\begin{array}{ccccc} (1_1, 6_1, 0_2) & (2_1, 5_1, 0_2) & (3_1, 4_1, 0_2) & (1_2, 6_2, 0_3) & (2_2, 5_2, 0_3) \\ (3_2, 4_2, 0_3) & (1_3, 6_3, 0_1) & (2_3, 5_3, 0_1) & (3_3, 4_3, 0_1) & (0_1, 0_2, 0_3) \end{array}$$

$\bmod (7, 3)$ which Bose gives as the initial blocks of a *BIBD*(21, 3, 1). Without the last block of $(0_1, 0_2, 0_3)$ we have a *GBRD*(7, 3, 9; Z_3).

Example 2 The subscripts from Bose's [4, p373, example (i)] *BIBD*(9, 3, 1) give a *GBRD*(3, 3, 9; Z_3) (the *GBRD* is the 3×9 array):

$$\begin{bmatrix} 2 & 1 & 1 & 3 & 2 & 2 & 1 & 3 & 3 \\ 1 & 2 & 1 & 2 & 3 & 2 & 3 & 1 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 1 \end{bmatrix}$$

Clearly, the differences from a single starting block also suffice to give all the differences, and hence we have a *GBRD*(3, 3, 3; Z_3).

Example 3 Consider the initial sets

$$(0_0, 1_0, 4_1) \text{ twice, } (\infty, 0_0, 3_0), (\infty, 0_1, 3_1) \bmod(5, 2)$$

To develop the subscripts we note

$$0^{-1} = 0, \quad 1^{-1} = 1, \quad 0 \cdot 1^{-1} = 1 \cdot 0^{-1} = 1, \quad 0 \cdot 0^{-1} = 1 \cdot 1^{-1} = 0.$$

We now note that the set $(\infty, 0_0, 3_0)$ means the differences ∞_0 and $-\infty_0$ each occur twice. The set $(\infty, 0_1, 3_1)$ means the differences ∞_1 and $-\infty_1$ also each occur twice.

When we check the non- ∞ differences we get

$$\begin{array}{llll} 0_0 - 1_0 = 4_0, & 0_0 - 4_1 = 1_1, & 1_0 - 4_1 = 2_1, & \text{twice,} \\ 1_0 - 0_0 = 1_0, & 4_1 - 0_0 = 4_1, & 4_1 - 1_0 = 3_1, & \text{twice,} \\ \text{and} & 3_0 - 0_0 = 3_0, & 0_0 - 3_0 = 2_0, & 3_1 - 0_1 = 3_0, \quad 0_1 - 3_1 = 2_0. \end{array}$$

As we have each non-zero difference occurring twice we have a $GBRD(6, 3, 4; Z_2)$. Its incidence matrix is

0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
.	1	1	1	1	1	1	1	1	1	1
0	1	.	.	0	0	1	.	.	0	0	.	0	.	.	1	.	1	.	.
0	0	1	.	.	0	0	1	.	.	.	0	.	0	.	.	1	.	1	.
.	0	0	1	.	.	0	0	1	.	.	.	0	.	0	.	.	1	.	1
.	.	0	0	1	.	.	0	0	1	0	.	.	0	.	1	.	.	1	.
1	.	.	0	0	1	.	.	0	0	.	0	.	.	0	.	1	.	.	1

where 0, 1 are elements of the group, Z_2 , and \cdot means the zero of the group ring so $\cdot - 1 = \cdot$, and $\cdot - 2 = \cdot$.

To change these sets into starting blocks for a $BIBD$ we would also need to have the difference 0_1 occur twice. This can be done but not in a straight forward way.

This clearly shows that Bhaskar Rao designs are not another representation of $BIBDs$. \square

Example 4 To illustrate how a design developed from blocks using differences can be shown to be a $BIBD$ or $GBRD$ we consider the initial blocks

$$(0_0, 1_0, 2_0) (0_0, 1_1, 2_1) (0_1, 2_0, 3_1) (0_0, 2_1, 4_1) (0_0, 2_0, 0_1) (0_0, 0_1, 4_1) \pmod{5, -}$$

So the differences are

$$\begin{array}{cccccc} 1_0 & 2_0 & 1_0 & 4_0 & 3_0 & 4_0 \\ 1_0 & 1_1 & 2_1 & 4_0 & 4_1 & 3_1 \\ 1_1 & 2_1 & 3_0 & 4_1 & 3_1 & 2_0 \\ 2_1 & 4_1 & 2_0 & 3_1 & 1_1 & 3_0 \\ 2_0 & 0_1 & 3_1 & 3_0 & 0_1 & 2_1 \\ 4_0 & 0_1 & 4_1 & 1_0 & 0_1 & 1_1 \end{array}$$

This means each of the differences $0_1, 1_0, 1_1, 2_0, 2_1, 3_0, 3_1, 4_0, 4_1$ occurs 4 times, that is 2 times corresponding to each difference where the direction it is taken in is considered. Hence we have, after developing the initial blocks modulo 5, exactly Bose's example (iv) [4, p370] for a $BIBD(10, 3, 2)$.

1	1	1	0	0	1	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0
0	1	1	1	0	0	1	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0
0	0	1	1	1	0	0	1	0	0	0	0	0	0	1	0	0	0	0	1	0	0
1	0	0	1	1	0	0	0	1	0	1	0	0	0	0	1	0	1	0	0	1	0
1	1	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	1	0	0	1	0
0	0	0	0	0	0	1	1	0	0	1	0	0	1	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	1	1	0	0	1	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	0	0	1	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	0	0	0	1	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	0	0	0	0	0	1	1

It has GBRD type incidence matrix for the last three sets of

$$\begin{array}{cccc|cccc|cccc} 0 & & 1 & . & 1 & 01 & . & 0 & . & . & 01 & . & . & . & . & 1 \\ 1 & 0 & . & 1 & . & . & 01 & . & 0 & . & 1 & 01 & . & . & . & . \\ . & 1 & 0 & . & 1 & . & . & 01 & . & 0 & . & 1 & 01 & . & . & . \\ 1 & . & 1 & 0 & . & 0 & . & . & 01 & . & . & . & 1 & 01 & . & . \\ . & 1 & . & 1 & 0 & . & 0 & . & . & 01 & . & . & . & 1 & 01 & . \end{array}$$

It is not a $GBRD$ as the difference 0_1 occurs 4 times. This is represented by the “01” in the above incidence matrix. However the first four blocks give a $GBRD(6, 3, 6; Z_2)$

$$(0_0, 1_0, 2_0) (0_0, 1_1, 2_1) (0_1, 2_0, 3_1) (0_0, 2_1, 4_1) \pmod{5, Z_2},$$

which has incidence matrix:

$$\begin{array}{ccccccccc} 0 & 0 & 0 & . & . & 0 & 1 & 1 & . & . & 0 & . & 0 & 1 & . & 0 & . & 1 & . & 1 \\ . & 0 & 0 & 0 & . & . & 0 & 1 & 1 & . & . & 0 & . & 0 & 1 & 1 & 0 & . & 1 & . \\ . & . & 0 & 0 & 0 & . & . & 0 & 1 & 1 & 1 & . & 0 & . & 0 & . & 1 & 0 & . & 1 \\ 0 & . & . & 0 & 0 & 1 & . & . & 0 & 1 & 0 & 1 & . & 0 & . & 1 & . & 1 & 0 & . \\ 0 & 0 & . & . & 0 & 1 & 1 & . & . & 0 & . & 0 & 1 & . & 0 & . & 1 & . & 1 & 0 \end{array}$$

Example 5 David Glynn [12] has found the only $GW(v, k, G)$ known to the author where G is not an abelian group. Consider the multiplication table for S_3

	1	2	3	4	5	6	
1	1	2	3	4	5	6	$1 \leftrightarrow e$
2	2	3	1	5	6	4	$2 \leftrightarrow (123)(456)$
3	3	1	2	6	4	5	$3 \leftrightarrow (132)(465)$
4	4	6	5	1	3	2	$4 \leftrightarrow (14)(26)(35)$
5	5	4	6	2	1	3	$5 \leftrightarrow (15)(24)(36)$
6	6	5	4	3	2	1	$6 \leftrightarrow (16)(25)(34)$

Then the circulant matrix

$$\begin{bmatrix} 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 \\ 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 \\ 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 \\ 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 \\ 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 \\ 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 \\ 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 \\ 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 \\ 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 \\ 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 \\ 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 \\ 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 \\ 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 \end{bmatrix}$$

is a generalized weighing matrix $GW(13, 9; S_3)$.

In set notation this can be written as

$$(1_5 \ 2_1 \ 3_4 \ 6_1 \ 7_1 \ 8_6 \ 9_5 \ 10_6 \ 12_4) \pmod{(13, S_3)}$$

There are two inequivalent circulant $GW(13, 9; Z_2)$ (see Seberry and Wehrhahn [23]) and a total of eight inequivalent $GW(13, 9; Z_2)$ (Ohmori [15]) but the existence of $GW(13, 9; G)$ for $G = Z_3$ or Z_6 is not yet resolved.

Another interesting possibility occurs if there are parallel classes associated with a subdesign. For example the $GBRD(13, 9, 6; S_3)$ given by developing the above block mod (13) can be embedded in an $SBIBD(91, 10, 1)$ which also has an $SBIBD(13, 4, 1)$ embedded (see the construction in [19]).

□

In this paper we show that $BIBD(v, b, r, k, \lambda)$, where $v = pq$ or $pq + 1$, when written in the notation of Bose's method of differences may often be used to find generalized Bhaskar Rao designs $GBRD(p, b', r', k, \lambda; G)$ where G is a group of order q .

Theorem 1 *Let G be a group. A $GBRD(v, k, \lambda; G)$, where $|G|$ is λ , is equivalent to a $BIBD(v\lambda, k, 1)$ when both are written using starting blocks developed mod(λ). (The most common case has G a cyclic group but it is possible to develop a theory with other groups.)*

Proof. Suppose we have initial blocks

$$(x_{a_i}^{(i)} \ y_{b_i}^{(i)} \ \dots \ z_{c_i}^{(i)})$$

for the $GBRD$. Then the equivalent starting blocks for the $BIBD$ are:

$$(x_{a_i}^{(i)}, \dots, z_{c_i}^{(i)}) \ (x_{a_i+1}^{(i)}, \dots, z_{c_i+1}^{(i)}) \ (x_{a_i+k-1}^{(i)}, \dots, z_{c_i+k-1}^{(i)}) \ (0_0, 0_1, \dots, 0_{k-1}) \ \text{mod}(v, \lambda).$$

The differences of the type $(x_{a_i}^{(i)} - y_{b_i}^{(i)})$, $a_i = 0, \dots, \lambda$ yield the product elements of type $(x^{(i)} - y^{(i)})_{a_i a_i^{-1}}$ which must occur once each for the $GBRD$ and once each arising from each a_i for the $BIBD$.

The differences of type $(x_{a_i}^{(i)} - y_{b_i}^{(i)})$, $a_i = 0, \dots, \lambda$, $a_i \neq b_i$, yield the product elements of type $(x^{(i)} - y^{(i)})_{a_i b_i^{-1}}$ which must occur equally often for the $GBRD$ and hence occur equally often for the $BIBD$. The block with zeros completes the differences for the $BIBD$. \square

See the case $k = 3$ in the next section for an example of how this theorem works.

When we have a $BIBD((k-1)v+1, k, k-1)$ developed mod $(v, k-1)$ with a special element ∞ we seek to form a $GBRD(v, k, (k-1)^2; G)$ where G is a group of order $k-1$. Again the converse, that the $GBRD$ in this case gives a $BIBD$, has been observed previously (see [19]).

Theorem 2 *Let G be a group. If the initial blocks*

$$(x_{a_1}^{(1)}, x_{a_2}^{(2)}, \dots, x_{a_k}^{(k)}) \ (y_{b_1}^{(1)}, y_{b_2}^{(2)}, \dots, y_{b_k}^{(k)}) \dots (z_{c_1}^{(1)}, z_{c_2}^{(2)}, \dots, z_{c_k}^{(k)})$$

give a $GBRD(v, k, k-1; G)$ where $|G| = k-1$, then the initial blocks

$$(x_{a_1+i}^{(1)}, x_{a_2+i}^{(2)}, \dots, x_{a_k+i}^{(k)}) \ (y_{b_1+i}^{(1)}, y_{b_2+i}^{(2)}, \dots, y_{b_k+i}^{(k)}) \dots (z_{c_1+i}^{(1)}, z_{c_2+i}^{(2)}, \dots, z_{c_k+i}^{(k)}) \ (\infty, 0_1, 0_2, \dots, 0_{k-1})$$

$i = 0, \dots, k-2$, give a $BIBD((k-1)v+1, k, 1)$, where $i = 0, 1, \dots, k-1$. The converse is also true. (The most common case has G a cyclic group but it is possible to develop a theory with other groups.)

Proof. The proof follows by noticing that the mixed non-zero differences occur the same number of times. The extra block makes sure the zero elements occur the appropriate number of times. \square

See the case $k = 4$ in the next section for an example of how this theorem works.

Comment. In some sense Bose's theorem implies that the subscripts are from a cyclic group. However the formulation in terms of $GBRD$ indicates clearly that any group will suffice.

The construction with non-cyclic groups can be visualized by replacing the 0 element of the $GBRD$ by a $(k-1) \times (k-1)$ zero matrix and the other elements by their right (left) regular matrix representation. Finally, the extra $k-1$ blocks corresponding to $(\infty \ 0_1 \ 0_2 \dots 0_{k-1})$ are added.

We now give some examples of these methods. The *GBRD* are new.

k=3: To illustrate

$$(0_1, 1_0, 4_0), (0_1, 2_0, 3_0) \bmod(5, Z_3)$$

give a *GBRD*(5, 3, 3; Z_3). The blocks

$$\begin{array}{lll} (0_1, 1_0, 4_0) & (0_1, 2_0, 3_0) & (0_0, 0_2, 0_3) \\ (0_2, 1_1, 4_1) & (0_2, 2_1, 3_1) & \\ (0_0, 1_2, 4_2) & (0_0, 2_2, 3_2) & \end{array} \bmod(5, 3)$$

give a *BIBD*(15, 3, 1). That the *GBRD* in this case gives a *BIBD*, has been observed previously (see [19]).

In Lemma 3 this method is applied to *BIBD*(3*v*, 3, 1) and *GBRD*(*v*, 3, 3; Z_3) for all odd *v*.

k=4: The initial blocks for the *BIBD*(28, 4, 1), *i* = 0, 1, 2, are

$$[(21)_i, (01)_i, (12)_{i+1}, (10)_{i+1}], [(20)_i, (02)_i, (22)_{i+1}, (00)_{i+1}] \text{ and } [\infty, (00)_0, (00)_1, (00)_2].$$

The first two blocks, with *i* = 0, give a *GBRD*(9, 4, 3; Z_3). The elements work modulo (3,3) and suffixes modulo 3.

k=5: The initial blocks of a *BIBD*(45, 5, 1) are

$$\begin{array}{l} [(01)_0, (02)_0, (10)_2, (20)_2, (00)_1], \\ [(21)_0, (12)_0, (22)_2, (11)_2, (00)_1] \\ \text{and } [(00)_0, (00)_1, (00)_2, (00)_3, (00)_4]. \end{array}$$

The first two blocks give a *GBRD*(9, 5, 5; Z_5) which is new. The elements work modulo (3,3) and suffixes modulo 5.

k=6: Denniston [10] has given a *BIBD*(66, 6, 1) design. This design is rewritten in a convenient way as follows:

$$\begin{array}{ll} (0_3, 2_3, 3_0, 5_2, 6_2, 10_0), & (0_3, 1_4, 4_0, 6_0, 7_1, 11_2), \\ (0_4, 1_4, 3_2, 9_2, 11_1, 12_2), & (0_3, 3_4, 5_1, 7_0, 8_2, 12_1), \\ (0_4, 1_0, 3_4, 7_2, 9_0, 10_0), & (2_3, 4_4, 5_3, 6_3, 10_4, 12_4), \\ (0_4, 1_1, 3_1, 4_1, 8_0, 9_4), & (0_3, 2_0, 4_1, 5_0, 6_3, 9_1), \\ (0_3, 1_2, 2_1, 5_3, 6_1, 12_2), & (0_3, 2_2, 8_1, 9_4, 10_2, 11_0), \\ (\infty, 0_0, 0_1, 0_2, 0_3, 0_4), & \end{array}$$

all blocks developed mod(13, 5). We note that the first 10 of these blocks also give a *GBRD*(13, 6, 25; Z_5). If we knew how to sign the blocks {2 5 6 7 8 11} and {1 3 4 9 10 12}, which develop mod(13) to give a *BIBD*(13, 6, 5), we would have a *GBRD*(13, 6, 5; Z_5).

k=7: The following three sets of initial blocks from Bhat-Nayak and Kane [1, 2] give initial blocks for a *BIBD*(91, 7, 1):

Solution 1:

$$\begin{aligned} & [6_0, 7_0, 11_1, 2_1, 8_5, 5_5, 0_2], \\ & [10_1, 3_1, 9_0, 4_0, 12_5, 1_5, 0_2] \\ \text{and } & [0_0, 0_1, 0_2, 0_3, 0_4, 0_5, 0_6]. \end{aligned}$$

Solution 2:

$$\begin{aligned} & [6_4, 7_4, 11_3, 2_3, 8_6, 5_6, 0_2], \\ & [10_3, 3_3, 9_4, 4_4, 12_6, 1_6, 0_2] \\ \text{and } & [0_0, 0_1, 0_2, 0_3, 0_4, 0_5, 0_6]. \end{aligned}$$

Solution 3:

$$\begin{aligned} & [6_4, 7_4, 11_3, 2_3, 8_6, 5_6, 0_2], \\ & [10_1, 3_1, 9_0, 4_0, 12_5, 1_5, 0_2] \\ \text{and } & [0_0, 0_1, 0_2, 0_3, 0_4, 0_5, 0_6]. \end{aligned}$$

In each case the first two blocks mod $(13, 7)$ are initial blocks for a $GBRD(13, 7, 7; Z_7)$. These are new.

One typical result illustrates many possible theorems arising from constructions for $BIBD$ using Bose's method of differences.

Lemma 3 *The starting blocks*

$$(1_0, 2t_0, 0_1) \ (2_0, (2t-1)_0, 0_1) \ \dots \ (t_0, (t+1)_0, 0_1) \ (0_0, 0_1, 0_2) \ \text{mod } (2t+1, 3)$$

were used by Bose [4, p373, equation 4.22] to form a $BIBD(3(2t+1), 3, 1)$. The starting blocks

$$(1_0, 2t_0, 0_1) \ (2_0, (2t-1)_0, 0_1) \ \dots \ (t_0, (t+1)_0, 0_1) \ \text{mod } (2t+1, Z_3)$$

form a $GBRD(2t+1, 3, 3; Z_3)$.

Proof. A simple check shows all the differences for the $GBRD$, $\{1, \dots, 2t\}$, are obtained with each subscript 0, 1, 2. This immediately gives the $GBRD(2t+1, 3, 3; Z_3)$.

It remains to notice that the starting blocks for the $BIBD$ are the same as for the $GBRD$ with the exception of the last. This means we have the required number of pairs both with the subscripts i and $i, i = 0, 1, 2$ and with cross subscripts i and $j, i \neq j, i, j = 0, 1, 2$. This gives the result after noting the block $(0_0, 0_1, 0_2)$ guarantees the remaining differences which are needed.

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